Lect21-0405b Connect

It is easier and more intuitive to start with disconnected.

A space $(X, \mathcal{T})$ is disconnected if

$$\exists\ \underbrace{U, V \in \mathcal{T}}_{\text{particularly important}},\ U, V \neq \phi,\ \underbrace{U \cap V = \phi,\ U \cup V = X}_{U = X \setminus V,\ V = X \setminus U}.$$

$$\therefore \left.\begin{matrix} V, X\setminus V \\ X \setminus U, U \end{matrix}\right\} \in \mathcal{T}$$

X is disconnected $\Leftrightarrow \exists\ \phi \neq U, V \subsetneqq X$ such that U, V are both open and closed.

Qu Write the negation of disconnected

There may be several ways to write it.

* If $\phi \neq U, V \subsetneqq X$ then $U \notin \mathcal{T}$ or $V \notin \mathcal{T}$ or $X \setminus U \notin \mathcal{T}$ or $X \setminus V \notin \mathcal{T}$
* If $\phi \neq U, V \subsetneqq X$ and $U, V \in \mathcal{T}$ then $X \setminus U$ or $X \setminus V \notin \mathcal{T}$

Definition (useful in doing proof)

$(X, \mathcal{T})$ is connected if $\forall\ U \subset X$ that is both open and closed in X, $U = \phi$ or $U = X$.

Note that no need to mention V in above

Example $X = Y \cup G \subset \mathbb{R}^2$ where

$$Y = \{(0,y) \in \mathbb{R}^2 : y \in \mathbb{R}\}$$

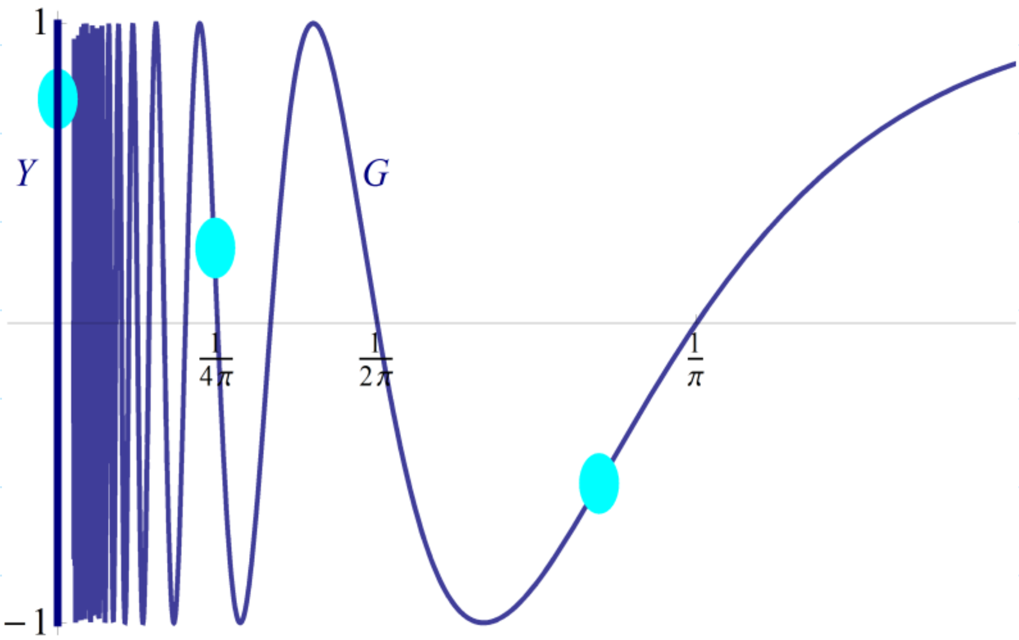
$$G = \{(x, \sin\tfrac{1}{x}) \in \mathbb{R}^2 : x > 0\}$$

X is a typical example of connected space.

Qu. How do we show it?

Let $U \subset X$ be both open and closed

Try to prove that $U = \phi$ or $U = X$



For simplicity, we accept that Y and G are connected. Consider $U \cap Y$ and $U \cap G$.

They are both open & closed in Y and G

$\therefore$ $U \cap Y = Y$ and $U \cap G = \phi$

or $U \cap Y = \phi$ and $U \cap G = G$

For the case $U \cap Y = Y$ and $U \cap G = \phi$

$\exists$ open set $W \in \mathcal{J}_{\mathbb{R}^2}$ such that

$$W \cap Y = Y \quad \text{and} \quad W \cap G = \phi$$

In particular, $W \supset \{0\} \times [-1,1]$ and $W \cap G = \phi$

Using compactness of $[-1,1]$, $\exists\ \delta > 0$

$$\{0\} \times [-1,1] \subset (-\delta, \delta) \times [-1,1] \subset W$$

However $(-\delta, \delta) \times [-1,1] \cap G \neq \phi$

For the case of $U \cap Y = \phi$ and $U \cap G = G$

We may use $(X \setminus U) \cap Y = Y$ and $(X \setminus U) \cap G = \phi$

and the above argument.

Or, we may take a sequence $(\frac{\pi}{n}, 0) \in U \cap G \subset U$

The sequence $(\frac{\pi}{n}, 0) \longrightarrow (0,0) \notin G = U \cap G$

Thus, $U \cap G$ is not closed in G

$\quad\ \ \shortparallel$

$U \cap X$ is not closed in X